1. The Boltzmann equation

The Boltzmann equation accounts for changes in the phase space number density. Physically, such changes may arise if particles (we will include photons in the term particles) are created or destroyed. If we have the particles completely contained in a region of space, then these two mechanisms are all that can change the total number in the region. However, due to the motion of the particles, their spatial distribution may change providing a mechanism by which the density may change. If for example the particles all coalesced into a small subregion of the container, the density would increase in that subregion and correspondingly decrease elsewhere. Such motion may be accompanied by a redistribution of energy, which is the basis of the convection mode of heat transport. It often happens that the phase space density becomes independent of time, This is referred to as the steady state condition.

Consideration is restricted to particles belonging to a specific energy group $E$ and directional group $\Omega$. A volume element is selected with area $\delta A$ oriented normal to the propagation direction at $r$. The axis of the element is of magnitude $\delta l$ and oriented normal to the area so that the second face is located at $r+\delta l$ where $\delta l = \delta l \cdot \Omega$. The change in the number of particles of interest in the element due to their motion or flow is

$$\delta N = \delta A \cdot [\tilde{\Phi}(r,E) - \tilde{\Phi}(r+\delta l,E)] d^3 E$$

where $\tilde{\Phi}(E,r)$ is the angular fluence spectrum. The second term may be expanded about $r$ in a Taylor's series to first order to give $\tilde{\Phi}(r,E) + \delta l \cdot \nabla \tilde{\Phi}(r,E)$. Substitution in Eq.(1) leads to the convection term

$$\delta \bar{n} = -\Omega \cdot \nabla \tilde{\Phi}(r,E)$$

where $\delta n = \delta N/\delta A \delta l$ is the density change. The convection term is the difference between the number of particles entering and leaving the region per unit volume and hence represents the net change in density. This change is manifested by the gradient of the field, an inherently mathematical property. The Boltzmann equation relates this mathematical behaviour to the other physical processes which must occur to cause the density change. These are of three distinct origins, interactions, production and particle decay. These will now be considered in turn.

Interactions of the particles belonging to the energy-directional group of interest with matter in the volume represent losses in the density. This is obvious if the interaction results in the absorption of the particle. It is also true in the case of scattering which alters the energy or direction of propagation. Scattering thus represents a transition to a different group. The interaction density is $\ddot{\mu}(r,E)\tilde{\Phi}(r,E)$ where $\ddot{\mu}(E,r)$ is the total interaction coefficient or linear attenuation coefficient at $r$. For a homogeneous region there is no dependence on position.

The scattering process may also increase the number of particles in the specified group by transitions from other groups. The interaction density for such in-scattering is
$$\delta n_{in} = \int \tilde{\mu}(E'|E)\tilde{\Phi}(r,E')d^3E'$$

the integral representing the sum over initial states or other groups. Again for simplicity, homogeneity is assumed, and a possible dependence of the interaction coefficient on position has not been indicated.

Sources of the radiation field are usually either radioisotopes emitting particles of interest or machine generated type conversions such as in X-ray production from electron radiative stopping. The source term is designated $
\tilde{Y}(r,E)\frac{d^6N_p}{d^3rd^3E}$ and represents the number of particles, $N_p$, produced per unit solid angle, energy and volume. The particles are emitted in the direction $\Omega$ with energy $E$, within the usual infinitesimal ranges.

If the particles are unstable then decay in flight represents an additional loss. If the mean life of the particle in the laboratory frame is $\tau$ then the transition probability per unit time is $1/\tau$. It follows therefore that the decay probability per unit path length is $1/v\tau$ where $v$ is the velocity of the particle with respect to the laboratory. The loss per unit volume due to decay is thus $\delta \tilde{\Phi}(r,E)/v\tau$. The lifetime in the laboratory frame is related to the particle lifetime (the mean life in the rest frame) by

$$\tau = \tau_0/\sqrt{1-v^2/c^2}$$

Introducing a mean free path for decay $\lambda_s=p\tau_0/m_o$, where $p$ is the momentum and $m_o$ is the rest mass, the density loss due to decay is $\Phi(r,E)\lambda_s^{-1}$.

The Boltzmann equation is then obtained by balancing the three loss terms due to convection, interactions and decay with the production and in-scattering to give

$$\Omega \cdot \nabla \tilde{\Phi} + (\tilde{\mu} + \lambda_s^{-1})\tilde{\Phi} = \tilde{Y} + \int \tilde{\mu}(E'|E)\tilde{\Phi}(r,E')d^3E'$$

An alternative derivation leading to the time dependent equation relevant when steady state conditions do not apply is as follows. The rate of change of density is the difference between the creation rate and the destruction rate. The former is the sum of the production rate and the in-scattering rate. The latter is the sum of the decay rate and interaction rate.

$$\frac{d\tilde{n}}{dt} = \frac{\partial\tilde{n}}{\partial t} + \sum_{i=1}^{3} \frac{\partial\tilde{n}}{\partial x_i} \frac{dx_i}{dt} = \frac{\partial\tilde{n}}{\partial t} + \Omega \cdot \nabla \tilde{\Phi} + \tilde{\zeta} + \int \tilde{\mu}(E'|E)\tilde{\Phi}(r,E')d^3E' - (\tilde{\mu} + \lambda_s^{-1})\tilde{\Phi}$$

where the spatial time derivatives are components of the velocity vector $v=\nu\Omega$ and $\tilde{\phi} = \tilde{n}v$

The quantity $\tilde{\zeta}(r,E,t)$ is the source production rate density. $\frac{d^6N_p}{d^3rd^3t}$. Rearranging gives the time dependent transport equation:

$$(NOTES 8) \ page 2$$
2. The continuous slowing down approximation

The in-scattering term in Equ.(5) may be transformed somewhat in the case where only small energy losses occur. Interactions are divided into elastic for which only angular changes occur and inelastic collisions in which only energy losses occur in this approximation. The latter are represented by a stopping power.

It is convenient to use the energy loss \( \varepsilon \) as the independent variable. The differential coefficient for inelastic scattering can then be written

\[ \mu (\varepsilon | E + \varepsilon) \] . The total interaction cross-section can be written

\[ \mu_i (\varepsilon | E) = \mu_a + \mu_e + \mu_i \] where \( \mu_i \) is the elastic scattering contribution. Charged particle absorption can only occur if nuclear reactions are induced, so \( \mu_a \) is generally quite small. The Boltzmann equation can then be written

\[ \Omega \cdot \nabla \tilde{\Phi} + (\tilde{\mu} + \lambda_y^{-1}) \tilde{\Phi} = \tilde{\Phi} + \int \tilde{\mu}_e (\Omega, \Omega') \tilde{\Phi}(r, E', \Omega') d\Omega' \]

\[ + \int \tilde{\mu}_i (\varepsilon | E + \varepsilon) \Phi(r, E + \varepsilon, \Omega) d\varepsilon - \int \tilde{\mu}_i (\varepsilon | E) \Phi(r, E, \Omega) d\varepsilon \] (8)

The last term is the expanded form of \( \mu_i \tilde{\Phi}(r, E, \Omega) \) and \( \tilde{\mu} = \tilde{\mu}_a + \tilde{\mu}_e \). If the third expression on the right hand side is expanded in Taylor's series to first order, and the difference between this result and the last term evaluated, the equation becomes

\[ \Omega \cdot \nabla \tilde{\Phi} + (\tilde{\mu} + \lambda_y^{-1}) \tilde{\Phi} = \tilde{\Phi} + \int \tilde{\mu}_e (\Omega, \Omega') \tilde{\Phi}(r, E, \Omega') d\Omega' \]

\[ + \int \tilde{\mu}_i (\varepsilon | E + \varepsilon) \Phi(r, E + \varepsilon, \Omega) d\varepsilon - \int \tilde{\mu}_i (\varepsilon | E) \Phi(r, E, \Omega) d\varepsilon \]

\[ = \tilde{\Phi} + \int \tilde{\mu}_e (\Omega, \Omega') \tilde{\Phi}(r, E, \Omega') d\Omega' \]

\[ + \int \frac{\partial}{\partial E} (S(E) \tilde{\Phi}(r, E, \Omega)) d\Omega' \] (9)

The last term may be abbreviated to \( \frac{\partial}{\partial E} (S \tilde{\Phi}) \), and is referred to as the slowing-down term. The product

\[ S \tilde{\Phi} \] is sometimes called the slowing-down density. An intuitive interpretation of the slowing-down term is as follows.

Physically two distinct phenomena occur during transport of particles distributed in energy. In the first place there is an average translation in energy so that the spectrum shifts downward. Secondly since the

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stopping power depends on energy different energy groups lose energy at different rates so that the spectrum is distorted. Consider the behaviour of a group of particles with energies initially between \(E\) and \(E + dE\) after propagating a distance \(\delta s\). The energies of the boundaries of the group become \(E - S(E)\delta s\) and \(E + dE - S(E + dE)\delta s\) respectively so that the energy range over which the particles are now distributed is

\[dE(1 - \frac{\partial S}{\partial E})\]

Consider a volume element as in section 1 with particles at \(r\) with energy \(E\). Those particles leaving the volume with this energy at \(r\) will have energy \(E - S(E)\delta l\) at \(r + \delta l\). The change in the number of particles in the group due to particle flow can now be written

\[\delta N = \left[ \tilde{\Phi}(r, \Omega, E)dE - \tilde{\Phi}(r + \delta l, \Omega, E - S(E)\delta l)\cdot \frac{\partial S}{\partial E} \right] \delta A d\Omega\]

\[= \left[ \frac{\partial \tilde{\Phi}}{\partial E} \cdot S + \tilde{\Phi} \cdot \frac{\partial S}{\partial E} - \Omega \cdot \nabla \tilde{\Phi} \right] \delta A \delta l dEd\Omega\]

Therefore in the C.S.D.A. the convection term can be written

\[\delta \tilde{n} = \frac{\partial (S \tilde{\Phi})}{\partial E} - \Omega \cdot \nabla \tilde{\Phi}\]

An alternative derivation within the context of the time dependent equation arises if the energy variable is considered as a continuous function of time. Letting an element of path length be \(d\delta s\) so that \(d\delta s^2 = dx^2 + dy^2 + dz^2\) the total time derivative of the phase space density (ie angular number density spectrum) can be written

\[\frac{d\tilde{n}}{dt} = \frac{\partial \tilde{n}}{\partial t} + \Omega \cdot \nabla \tilde{\Phi} + \frac{\partial}{\partial \tilde{n}}\left( \tilde{n} \cdot dE \right)\]

\[= \frac{\partial \tilde{n}}{\partial t} + \Omega \cdot \nabla \tilde{\Phi} + \frac{\partial}{\partial E}\left( \tilde{n} \cdot dE \right)\]

\[= \frac{\partial \tilde{n}}{\partial t} + \Omega \cdot \nabla \tilde{\Phi} - \frac{\partial}{\partial E}(S\tilde{\Phi})\]

Using this time derivative in Eqn(6) results in the time dependent Boltzmann equation in the CSDA.

3. Streaming

In this the simplest case, all interactions vanish. If we restrict discussion for the moment to stable particles the steady state Boltzmann equation becomes

\[\Omega \cdot \nabla \tilde{\Phi} = \tilde{Y}(r,E)\]
In the absence of sources the above equation is a statement of the invariance of the angular fluence, or equivalently the irradiance, as discussed previously in section 3. The term on the left represents the rate of change in the fluence with distance along a line oriented in the direction $\Omega$. The line may be written $r'=r+s\Omega$. The problem then becomes a one-dimensional one in a coordinate system along this line, the Boltzmann equation becoming simply

$$\frac{d\Phi}{ds} = \tilde{Y} \quad (14)$$

with solution

$$\Phi(r,E) = \int_{-\infty}^{0} \tilde{Y}(r+s\Omega,E)ds \quad (15)$$

The geometry is indicated in Figure 1. Physically, radiation emitted from a point in the source indicated by the shaded region, in direction $\Omega$ continues in a straight line in that direction. If the line intersects the point of interest at $x$ then it contributes an amount $\tilde{Y}ds$ to the fluence. The fluence is calculated by “looking back” from the point of interest in the opposite direction to $\Omega$ and adding up all the above contributions.

A relatively simple case is that of an isotropic source distributed uniformly over a spherical region of radius $a$. Choose the origin of the coordinate system to be at the centre of the sphere. Then the angular source density spectrum satisfies

$$\tilde{Y} = \frac{1}{4\pi} \frac{3Q(E)}{4\pi a^3} \quad r' \leq a \quad (16)$$

$$= 0 \quad r' > a$$

where $Q(E)$ is the total number of particles emitted with energy $E$. It is convenient to locate the point at which the angular fluence spectrum is evaluated on the $z$-axis with $z=r$. The geometry is depicted in Fig.2 where it can be seen that the required line integral is the product of the constant angular source density spectrum with the chord length $\tilde{Y}(E)r$, where the constant source density is equal to the right side of Equ.(16).

The chord forms the base of two right angled triangles each with hypotenuse $a$. The common side opposite the hypotenuse is $r\sin\theta$. It is necessary to restrict the range of the angle $\theta$. Obviously if this angle becomes too large the line will not intersect the source at all and in this case the angular fluence spectrum vanishes. The condition for a non-vanishing flux is then $\sin\theta \leq a/r$. Note that $\theta$ is not the polar angle of the point $r$.

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When the above condition is satisfied the angular fluence spectrum is then given by

$$\Phi(r,E,\theta) = 2\tilde{Y}(E)\sqrt{a^2 - r^2 \sin^2 \theta}$$

The angular fluence is maximum in the direction $\theta=0$, in which case the chord length equals the diameter of the sphere, and decreases to zero as the angle increases to $\sin^{-1}a/r$.

Of particular interest is the limit $a \to 0$, in which the sphere approximates a point source. Then the source density becomes infinite, while at the same time the limiting angle becomes zero. The angular flux density is then a delta function having a non-zero value only corresponding to $\Omega=\nabla z$. The fluence spectrum is obtained by integration over solid angle. Using azimuthal symmetry

$$\Phi(r,E) = 4\pi \tilde{Y}(E) \int_0^{\sin^{-1}a/r} \sin \theta \sqrt{a^2 - r^2 \sin^2 \theta} d\theta$$

In the limit as a point source is approached the small angle approximation may be used and Eqn.(18) becomes

$$\Phi(r,E) \approx 4\pi \tilde{Y}(E) \int_0^{a/r} \sqrt{a^2 - r^2 \theta^2} d\theta$$

$$\frac{4\pi a^3}{3r^2} \tilde{Y}(E) = \frac{Q(E)}{4\pi r^2}$$

The final result is to be expected since at distance $r$ the $Q(E)$ emitted particles are uniformly distributed over a sphere of radius $r$.

Consider next the case in which the source is in the form of a disc of radius $a$ in the x-y plane so that

$$\tilde{Y}(E) = \frac{1}{4\pi} \frac{Q(E)}{\pi a^2} \delta(z)$$

Then, using $z = r \cdot s \cos \theta$ the angular fluence spectrum is

$$\tilde{\Phi}(r,E,\theta) = \frac{1}{4\pi} \frac{Q(E)}{\pi a^2} \delta(z) ds = \frac{1}{4\pi} \frac{Q(E)}{\pi a^2} \sec \theta \cdot \sec \theta \leq \sqrt{1 + a^2/r^2}$$

Figure 2 Uniform sphere

(Note: Figures and diagrams should be included if available.)
vanishing beyond the limit. Integration over solid angle gives

$$\Phi(r,E) = \frac{O(E)}{4\pi a^2} \ln(1 + a^2/r^2) \quad (21)$$

As before the angular fluence becomes singular as the disc radius decreases to zero and Eq.(21) reduces to the same result as Eq.(19).

If the particles are unstable, then the steady state Boltzmann equation becomes

$$\frac{d\tilde{\Phi}}{ds} - \frac{\tilde{\Phi}}{\lambda_s} = \tilde{Y}(r,E) \quad (22)$$

The general solution in this case can be written

$$\tilde{\Phi}(r,E) = \int_{-\infty}^{0} \tilde{Y}(r+s\Omega,E)e^{s/\lambda_s}ds \quad (23)$$

The exponential factor corrects for the loss of particles emitted from a point in the source a distance $|s|$ from the point of observation (remember $s$ is negative) due to decay. For the isotropic disc source the integral becomes

$$\tilde{\Phi}(r,E,\theta) = \frac{1}{4\pi} \frac{O(E)}{\pi a^2} \int_{-\infty}^{0} e^{s/\lambda_s} \delta(z)ds$$

$$= \frac{1}{4\pi} \frac{O(E)}{\pi a^2} [e^{-r\sec\theta/\lambda_s}] \sec\theta \sec\theta \sqrt{1 + a^2/r^2} \quad (24)$$

The effect of the exponential term is to produce a distribution peaked for $\theta=0$ when $r$ is significant with respect to $\lambda$. This is because at any other angle the path length from the origin of the particle created on the disc to the point of interest on the axis is greater than the normal distance, $r$. As a consequence a greater number of particles decay before reaching the observation point. For $r<<\lambda$ the decay is inconsequential, and the solution is approximately the same as Eq.(20), for stable particles. In the other extreme, $r>>\lambda$, the decay dominates and the angular fluence is essentially singular, being significant only for a very narrow range of directions near the normal to the disc. Solution for the spherical source requires tedious geometrical analysis.

Examination of the steady state Boltzmann equation shows that the above approach can be directly extended to the case of purely absorbing media. All that is necessary is to replace $1/\lambda_s$ by $\mu + 1/\lambda_s$. Of course this is physically no longer the same as streaming, if the latter is taken to mean radiative transport in vacuo.
However, in the absence of all scattering processes the particles once emitted continue to move in straight lines along the original direction of emission in the absorbing medium. This is sufficient for this approach. Possible examples where this approximation would be suitable are low energy photons propagating in a high Z material where the photoelectric cross section is dominant, and thermal neutron transport in a medium rich in boron or lithium, both of which have large \((n,\alpha)\) cross sections. Charged particle transport never really satisfies the requirements since the atomic processes, which dominate over nuclear interactions, are scattering. Here a different approximation, based on the CSDA is often adopted as discussed in the next section.

Since the transport equation is linear it is always possible to decompose the angular fluence into the sum of two terms, a direct and scattered term. The direct term is a solution to Equ.(22), augmented by the inclusion of the term representing interaction losses, \(\hat{\mu}\hat{\Phi}\). The scattered term is a solution to Equ.(5) with \(Y=0\). This equation can in principle be solved by iteration.

4. The straight ahead approximation

The straight ahead approximation is useful in describing transport when beams of particles which undergo little angular deviations are considered. This situation arises with very high energy \(\gamma\)-rays since the Compton scattering process becomes forward directed. It also occurs to a reasonable approximation for heavy charged particles. For this reason the approximation has been used in cosmic ray and space science studies. If the beam direction is taken along the \(x\) axis then since no scattering occurs in a one-dimensional description the fluence spectrum depends only on \(E\) and \(x\) and the Boltzmann equation becomes

\[
\frac{\partial \Phi}{\partial x} + \left(\mu_a + \lambda_c^{-1}\right)\Phi = \frac{\partial (S\Phi)}{\partial E} + Y
\]

In the above the interaction coefficient now refers only to absorptive processes. For heavy charged particles these occur for nuclear reactions and if the energy is insufficient these are excluded. For a source free region in this case, with stable particles the equation reduces to

\[
\frac{\partial \Phi}{\partial x} = \frac{\partial (S\Phi)}{\partial E}
\]

Notice that if the stopping power were constant the solution would be \(\Phi(E,x) = \Phi(E+Sx,0)\). This is consistent with a translation of the spectrum downward an amount \(Sx\). Such an approximation necessarily breaks down when parts of the spectrum pass through zero energy.

An equation may be developed for the fluence at \(x\) which is related to the fluence spectrum by

\[
\hat{\Phi}(x) = \int_0^\infty \Phi(E,x)\,dE
\]

The integral of the slowing down term over energy reduces to the difference between the slowing down density at 0 and \(\infty\) and hence vanishes. Defining the integrated source term \(Y(x)\) as the source integrated over all energies leads to
\[
\frac{d\hat{\Phi}}{dx} + \int_0^\infty (\mu_a + \lambda_\chi^{-1})\Phi(E,x)dE = Y(x) \tag{28}
\]

The second term is problematic in general unless both the interaction coefficient and decay mean free path may be treated as independent of energy. This situation is unlikely so that discussion is often restricted to the case of a monoenergetic spectrum. If the initial energy is \(E_0\) then the singular flux spectrum may be represented as

\[
\Phi(E,x) = \Phi(x)\delta(E-E_0(x)) \tag{29}
\]

where the function \(\varepsilon\) satisfies

\[
x = \int_{E_0}^{E} \frac{dE}{S(E)} \tag{30}
\]

The equation for the total flux may now be written

\[
\frac{d\hat{\Phi}}{dx} + [\mu_a(x) + \lambda_\chi^{-1}(x)]\Phi = Y(x) \tag{31}
\]

The dependence of the interaction coefficient and the decay mean free path on position is implicit by virtue of their dependence on energy which in turn is a well-defined function of position in the C.S.D.A. Of course in the case of the interaction coefficient this dependence is exhibited by the cross section. If the medium is inhomogeneous so that the target density is a function of position then this explicit dependence must also be included. Similarly, the mean free path for decay will change as the energy decreases due to the change in the Lorentz contraction. If we restrict discussion to stable particles, then the procedure by which the spatial dependence of the absorption coefficient is obtained in homogeneous media follows form Eqn(29).

5. **Point sources**

The point source solution is generally important since any source distribution may be thought of as composed of an infinite number of infinitesimal point sources. For any linear phenomenon this leads to solution by linear superposition often referred to point kernel methodology.

Consider a photon source situated at the origin. The point source concept requires the introduction of an singular source density which vanishes everywhere except \(r=0\). The source strength density is then given by

\[
\tilde{Y}(r,E) = \tilde{Q}(E,\theta',\phi')\delta(r) \text{ cm}^3 \text{ MeV}^{-1}\text{Sr}^{-1}. \tag{32}
\]

Since the source is located at the origin the emission direction must be radial, and points from the origin in the direction defined by \(\theta'\) and \(\phi'\).

The angular fluence spectrum is now decomposed into a direct and scattered term according to
\[ \hat{\Phi}(E, \Omega, r) = \Phi_d(E, r) \delta(1 - \Omega \cdot \hat{r}) + \tilde{\Phi}_s(E, \Omega, r) \]  
(32)

The direct term is also singular since the photon direction is unique if the radiation propagates in a straight line from the source to the point of observation. The direction must be along the radius so that \( \Omega = \hat{r} \), the unit radial vector. Also because the direct term is singular the value in the radial direction is the fluence spectrum. The direct vector fluence spectrum has only a radial component given by \( \Phi_d = \Phi_d \hat{r} \).

The two components now satisfy
\[ \hat{r} \nabla \Phi_d + \mu \Phi_d = 0 \quad r > 0 \]  
(33)
and
\[ \Omega \cdot \nabla \tilde{\Phi}_s + \mu \tilde{\Phi}_s = \int \mu(E | E') \Phi(E') d^3 E' \]  
(34)
Making use of the identity \( \hat{r} \nabla \Phi = \nabla \hat{r} \Phi \) so only the radial component of the divergence is involved and Equ.(33) becomes
\[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \Phi_d) + \mu \Phi_d = 0 \quad , r > 0 \]  
(35)

The above equation has as its general solution
\[ \Phi_d(r, E) = G(\Omega, E) e^{-\mu r / r^2} \]  
(36)
Now the number of photons which have passed through the area element at \( r \) subtending solid angle \( d\Omega \) is
\[ dN_d = \Phi_d(r, E) r^2 d\Omega \]  
and
\[ \lim_{r \to 0} dN_d = \tilde{Q}(E, \Omega) d\Omega \]  
by definition of the source strength. From (36) the identification of \( G(\Omega, E) \) with the source term is made so that
\[ \Phi_d(r, E) = \tilde{Q}(E, \theta', \phi') e^{-\mu r / r^2} \]  
(37)

It is important to note that the quantity in Equ.(35) is the fluence at the position \( (r, \theta', \phi') \) and not the angular fluence which is in this case the product of the fluence with \( \delta(1 - \Omega \cdot \hat{r}) \). While the most general situation is an anisotropic source, the most common case is that of a radioactive point object in which case emission in all directions is equally likely and \( \tilde{Q} \) is independent of \( \Omega \). The angular source density is then related to the total source strength \( Q(E) \) by \( \tilde{Q}(E, \theta', \phi') = Q(E) / 4\pi \). This leads to the common result
\[ \Phi_d(r, E) = Q(E) e^{-\mu r / 4\pi r^2} \]  
(38)

The above result also follows from Eqn(21). For a point source at the origin, \( s \) can be taken as -\( r \), and the exponential taken outside the integral. The evaluation of the latter is given by Eqn(17). Of course it is necessary to replace \( 1/\kappa_s \) by \( \mu \).

The determination of the scattered flux is not so easy. One approach which is easily visualized but still difficult to implement is an iterative procedure referred to as the method of successive scatterings. The solution
to Eq.(32) obtained when the direct fluence is substituted for $\Phi$ in the scattering integral is the singly scattered fluence, $\Phi_1$. Doubly scattered fluence is now obtained when $\Phi_1$ is used in the integral and so on. This will be demonstrated more fully in the next section.

6. Equilibrium

Consider the situation which arises when the source density is independent of position. The implication is that the source is distributed uniformly throughout all space. This is of course unrealistic but may approximate situations in the interior of volumes with dimensions large compared to the mean free path or range. This would apply for example to the radiation field generated by $^{40}\text{K}$ dissolved in the oceans. A more common case is the charged particle radiation field arising from radioactivity uniformly distributed throughout an organ such as a kidney. This might occur following administration of a radiopharmaceutical in a diagnostic procedure or be a result of accidental ingestion. Such equilibrium conditions then are approximate descriptions of geophysical and astrophysical phenomena as well as situations in nuclear medicine and health physics.

From the geometrical viewpoint the equilibrium geometry is the antithesis of the point source. While the isotropic point source produces an extremely anisotropic field to a singular degree, symmetry requires that the field produced by the uniform distribution of isotropic sources must itself be isotropic. Of course this requirement would not exist for a distribution of anisotropic sources but such a situation is difficult to envisage. The following does not rely on restriction to isotropic sources however. In any case it is clear that the radiation field cannot depend upon position so that the convective term must vanish, ie. $\nabla \Phi=0$. Designating the fluence spectrum by $\Phi(E)$, obtained by integration of the angular fluence spectrum over solid angle, it is then possible to determine an appropriate relation by the same integration of each term in the Boltzmann equation.

The result is

$$\hat{\mu}(E)\Phi(E) = Y(E) + \int \mu(E|E')\Phi(E')dE' \quad (39)$$

For photon radiation fields the successive scattering approach is applicable. The equilibrium direct component satisfies

$$\Phi_d(E) = Y(E)/\hat{\mu}(E) \quad (40)$$

This same result can be easily obtained by point source methodology using as a differential source a spherical shell so that $dY = 4\pi r^2 dr$. Combining this with the result in Eq.(38) and integrating over $r$ leads to Eq.(40). The scattered fields then satisfy

$$\Phi_n(E) = \hat{\mu}(E)^{-1}\int \mu(E|E')\Phi_{n-1}(E')dE' \quad (41)$$

For charged particle fields Eq.(39) may be reinterpreted within the context of the continuous slowing down approximation without absorption by noting
\[ \int \mu(E|E')\Phi(E')dE' - \tilde{\mu}(E)\Phi(E) = \int [\mu(E|E')\Phi(E') - \mu(E'|E)\Phi(E)]dE' \]

\[ \frac{d(S\Phi)}{dE} \]

The resulting equilibrium equation now becomes

\[ \frac{d(S\Phi)}{dE} = -Y(E) \]

The above equation could also have been derived from the solid angle integration of Equ.(8). In so doing it is assumed that the particles are stable and that the loss term is solely due to scattering so that

\[ \tilde{\mu}\Phi(E) = \int \mu(\Omega'|\Omega)\tilde{\Phi}(E,\Omega)d\Omega' \]

Integration over the in-scattering term leads to

\[ \int \tilde{\mu}(\Omega'|\Omega)\tilde{\Phi}(E,\Omega)d\Omega' \]

This integral is obviously identical to that in Equ.(42) indicating that redirection through scattering is irrelevant to quantities such as the fluence which result from integration over all directions. Cancellation of these two terms in the integration of Equ.(8) then leads to Equ.(41). The equilibrium spectrum follows from energy integration of the latter giving

\[ \Phi(E) = S(E)^{-1} \int_{E}^{\infty} Y(E')dE' \]

In obtaining the above result it is understood that the source density vanishes at some maximum energy and the order of integration from the initial higher energy to the final energy at E has been reversed thus eliminating the negative relationship.

A more intuitive derivation of Eqn(43) can be developed for a monoenergetic source, density \( \tilde{Y} \). Since for any observation point, particles are emitted at random distances, then it is equally probable that at this point any position along the track will be observed. Thus the probability of observing the point between \( s \) and \( s+ds \) along the track is simply \( dW = ds/R \) where \( R \) is the total length of the track, ie the range. The energy pdf is then given by

\[ p(E) = \frac{dW}{dE} = \frac{1}{RS} \]

where \( S = -\frac{dE}{ds} \). As discussed in section 3, the fluence is the density of track length, so \( \Phi = R\tilde{Y} \).

The fluence spectrum is \( \Phi(E) = p(E)\Phi = \tilde{Y}/S(E) \)

7. **The diffusion approximation**

For a system in which the particles are in thermal equilibrium with the surroundings the scattering process
will on average not alter the particle energy. The differential interaction coefficient only involves a change in direction. The energy distribution is the Maxwell Boltzmann distribution and is stationary. This situation occurs for neutrons as well as stable entities such as molecules. For simplicity consider a system which has lateral and azimuthal symmetry so that the angular flux density will only depend on one spatial coordinate and one angular coordinate. The Boltzmann equation for an isotropic source under these restrictions may be written

\[ \omega \frac{\partial \tilde{\Phi}}{\partial z} + \mu \tilde{\Phi} = \bar{\gamma} + \int \tilde{\mu}(\Omega | \Omega') \tilde{\Phi}(z, \omega') d\Omega'. \] (46)

where \( \mu = \mu_a + \mu_e \). Since there is no energy change there is no intermediate integration and it is not necessary to indicate the total coefficients by a circumflex. In the above \( \omega \) is \( \cos \theta \) where \( \theta \) is the angle between \( \Omega \) and the \( z \)-axis. The angle through which the particle is scattered in the integrand on the right satisfies

\[ \omega_s = \cos \theta_s = \Omega' \cdot \Omega \] (47)

and the scattering co-efficient may be written \( \tilde{\mu}(\omega_s) \).

Further analysis proceeds by expanding the angular flux density and scattering coefficient in a series of Legendre polynomials according to

\[ \tilde{\Phi}(z, \omega) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} \Phi_\ell(z) P_\ell(\omega) \] (48)

and

\[ \tilde{\mu}(\omega_s) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi} \mu_\ell P_\ell(\omega_s) \] (49)

The Legendre polynomials satisfy the orthogonality condition

\[ \int P_\ell(\omega) P_\kappa(\omega) d\Omega = \frac{4\pi}{2\ell+1} \delta_{\ell\kappa} \] (50)

and the addition theorem

\[ P_\ell(\omega_s) = P_\ell(\omega) P_\ell(\omega') + F(\Omega, \Omega'). \] (51)

In terms of the angular coordinates of \( \Omega', \theta' \) and \( \varphi' \), the second function on the right side of Equ.(51) vanishes when integrated over the azimuthal angle \( \varphi' \). Using the expansions in Equ.(48) and (49) and the properties in (50) and (51) the scattering integral in Equ.(46) can be written

\[ \int \tilde{\mu}(\Omega | \Omega') \tilde{\Phi}(z, \omega') d\Omega' = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \mu_\ell P_\ell(\omega) \] (52)

This leads to the alternative form for Equ.(44) given by
\[
\sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\omega) \left[ P_\ell(\omega) \frac{\partial \Phi_\ell}{\partial z} + \mu \Phi_\ell - \mu_\ell \Phi_\ell \right] = \tilde{Y}
\]  \hspace{1cm} (53)

In obtaining the above result use has been made of the fact that \( P_\ell(\omega) = 0 \). Equ.(53) can be resolved into a series of coupled equations by again making use of the orthogonality conditions, although the first term in general poses special problems. The diffusion approximation results from terminating the expansion after the first two terms. Consider the result obtained by multiplying Equ.(53) through by \( P_0(\omega) = 1 \) and integrating over solid angle. Then, making use of orthogonality

\[
\frac{\partial \Phi_1}{\partial z} + (\mu - \mu_0) \Phi_0 = \tilde{Y}
\]  \hspace{1cm} (54)

Similarly a second equation is obtained by multiplying through by \( P_1(\omega) \) and integrating over solid angle. In this case however the first term needs special treatment. Since the cut-off is \( \ell = 1 \) the only contribution to the first term comes from the \( \ell = 0 \) term. The result is

\[
\frac{1}{3} \frac{\partial \Phi_0}{\partial z} + (\mu - \mu_1) \Phi_1 = 0
\]  \hspace{1cm} (55)

The diffusion coefficient is introduced through the definition

\[ D = 1/3(\mu - \mu_1) \]  \hspace{1cm} (56)

so that Equ.(55) becomes the Fick's law

\[
\Phi_1 = -D \frac{\partial \Phi_0}{\partial z}
\]  \hspace{1cm} (57)

while Equ.(54) becomes the diffusion equation

\[
D \frac{\partial^2 \Phi_0}{\partial z^2} = (\mu - \mu_0) \Phi_0 - \tilde{Y}.
\]  \hspace{1cm} (58)

This result may be generalized if a more detailed interpretation of the quantities introduced is made. The angular fluence rate and differential interaction coefficient expansion coefficients are given by

\[
\varphi_\ell(z) = \int \tilde{\varphi}(z,\omega) P_\ell(\omega) d\Omega
\]  \hspace{1cm} (59)

while

\[
\mu_\ell = \int \tilde{\mu}(\omega) P_\ell(\omega) d\Omega
\]  \hspace{1cm} (60)

From Equ.(59) it may immediately be seen that the \( \ell = 0 \) coefficient is the flux density while the \( \ell = 1 \) coefficient is
the \( z \) component of current density. From Equ.(60) it follows that the \( l=0 \) coefficient for the differential interaction coefficient is the linear attenuation coefficient for scattering. The difference between the total and scattering quantities appearing on the right side of Equ.(58) is the linear attenuation coefficient for absorption.

No particular designation is used for the quantity \( \mu_1 \), but it can be written \( \mu_1 = \langle \omega_s \rangle \mu_0 \) and the difference \( \mu_t = (1 - \langle \omega_s \rangle) \mu_0 + \mu_a \) is referred to as the transport coefficient.

In neutron physics it is customary to refer to the interaction coefficients as macroscopic cross sections with the appropriate alteration in symbols. Introducing this nomenclature and generalizing to three spatial dimensions, Fick's law in Equ.(57) is written

\[
J = -D \nabla \varphi
\] (61)

in terms of current density and fluence rate. Moreover the difference

\[
\Sigma - \Sigma_1 = \Sigma_a + \Sigma_0 - \Sigma_i
\] (62)

is the macroscopic transport cross section, \( \Sigma_1 \). This leads to the identification of the diffusion coefficient as one third of the transport mean free path

\[
D = 1/3 \Sigma_i = \lambda_t/3
\] (63)

The diffusion equation, Equ.(58) is then generalized to

\[
D \nabla^2 \varphi - \Sigma_a \varphi + \xi = 0.
\] (64)

For the simple 1-dimensional case of an infinite source in the x-y plane the solution to Equ.(64) takes the form

\[
\varphi(z) = \varphi(0) e^{-|z|/L_d}
\] (65)

where the thermal diffusion length satisfies

\[
L_d = \sqrt{\lambda_t \lambda_a/3}.
\] (66)

The constant in Eqn(65) is determined by requiring the normal current density component to equal half the source strength in the limit as \( z \) tends to zero.

The concept of the transport cross section deserves more consideration. Elastic scattering at very small angles leads to negligible energy and directional change and has therefore an insignificant effect on the transport of the radiation. In the extreme situation of a differential cross section strongly peaked at zero scattering angle, and no absorption, \( \langle \omega_s \rangle \) approaches one and \( \Sigma_a \) approaches zero. This is reflected in an enhancement of the transport mean free path and the diffusion length given in Equ.(66). Simply put, elastic scattering through an angle of zero is indistinguishable from no interaction at all. In the common situation of isotropic scattering \( \langle \omega_s \rangle = 0 \) and the scattering part of the transport cross section becomes identical with the elastic scattering cross section.
The point source solution to the diffusion equation in an infinite medium is given by

$$\Phi = Q e^{-r/L_d}$$ \hspace{1cm} (67)

where Q is the point source strength, and time integrated quantities have been used.

8. **Age theory**

For radiation not in thermal equilibrium, collisions result in a net energy loss and diffusion is combined with slowing down. In standard age theory absorption is neglected so that addition of a slowing down term to the diffusion equation results in

$$D \nabla^2 \phi + \frac{\partial}{\partial E} (S \phi) = 0$$ \hspace{1cm} (68)

It is convenient to introduce the slowing down density \( q = S \phi \) and the age, given by \( d\tau = -D dE/S \), so that the above becomes

$$\nabla^2 q - \frac{\partial q}{\partial \tau} = 0$$ \hspace{1cm} (69)

which is the standard thermal diffusion equation. Recalling that for neutrons \( S = \xi \sum_s E \) the age corresponding to slowing down from energy \( E_i \) to \( E_f \) is given by

$$\tau = \frac{1}{E_f} \int_{E_f}^{E_i} D dE \frac{D \cdot dE}{\xi \sum_s E}$$ \hspace{1cm} (70)

The interpretation of the age becomes somewhat clearer if the approximation is made that \( D \) and \( \sum_s \) are independent of energy. Then using mean-free paths the above may be written

$$\tau = \frac{\lambda_s}{3\xi} \ln(E_i/E_f).$$ \hspace{1cm} (71)

Note that the age has dimension length squared so that the square root is a characteristic length referred to as the fast diffusion length and given by

$$L_{f} = \sqrt{N\lambda_s \lambda_s/3}.$$ \hspace{1cm} (72)
The factor $N = \xi^{-1} \ln(E_i/E_f)$ is the average number of collisions required to reduce the energy from $E_i$ to $E_f$. The dependence of the length on $\sqrt{N}$ is characteristic of a random walk process such as diffusion. The fast diffusion length is proportional to the geometric mean of the scattering and transport mean free paths. The solution for a point source is

$$q(r,\tau) = (4\pi \tau)^{-3/2} e^{-r^2/4\tau} \quad (73)$$

8. **The Fermi-Eyges Equation**

The Fermi-Eyges equation was derived independently to describe the behaviour of charged particle beams. Due to scattering the initially highly collimated beam acquires an angular distribution as it penetrates into the medium. As a consequence, a narrow beam begins to spread laterally. The equation, which ignores slowing down, is derived from the Boltzmann equation in the following. It is assumed that the incident beam is in the $z$-direction.

Define projected angles $\theta_x$ and $\theta_y$ through the transformation

$$\tan \theta_x = \Omega_x/\Omega_z = \tan \theta \cos \phi$$
$$\tan \theta_y = \Omega_y/\Omega_z = \tan \theta \sin \phi$$

The Fermi-Eyges equation can be obtained by employing the small angle approximation $\theta << 1$ in which case $\Omega_z = 1$ and

$$\theta_x = \Omega_x = \theta \cos \phi$$
$$\theta_y = \Omega_y = \theta \sin \phi$$

while the solid angle is approximately

$$d\Omega = \theta d\theta d\phi = d\theta_x d\theta_y$$

The Boltzmann equation neglecting energy loss is

$$\Omega \cdot \nabla \tilde{\phi} = \int [\tilde{\mu}(\Omega|\Omega') \tilde{\phi}(\Omega') - \tilde{\mu}(\Omega'|\Omega) \tilde{\phi}(\Omega)] d\Omega' \quad (74)$$

If one introduces the vector representing the change in projected angles following an interaction through $\Omega' = \Omega + \xi$ then on the basis of symmetry one can write $\tilde{\mu}(\Omega|\Omega') = \tilde{\mu}(\Omega'|\Omega) = \tilde{\mu}(\xi)$ and $\Omega' = d\xi_x d\xi_y$

The Boltzmann equation may now be written

$$\frac{\partial \tilde{\phi}}{\partial z} + \theta_x \frac{\partial \tilde{\phi}}{\partial x} + \theta_y \frac{\partial \tilde{\phi}}{\partial y} = \int [\tilde{\mu}(\xi) \tilde{\phi}(\Omega + \xi) - \tilde{\phi}(\Omega)] d\xi_x d\xi_y \quad (75)$$
Expanding the bracketed term to second order and recognizing that the first moments of $\mu(\xi)$ must vanish gives

$$
\frac{\partial \phi}{\partial z} + \theta_x \frac{\partial \phi}{\partial x} + \theta_y \frac{\partial \phi}{\partial y} = \frac{1}{2} \int [\xi^2 \mu(\xi) d^2 \xi] \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} \left( \frac{\partial^2 \phi}{\partial y^2} \right) \right] + \frac{1}{2} \int \xi^2 \mu(\xi) d^2 \xi \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} \left( \frac{\partial^2 \phi}{\partial y^2} \right) \right] \tag{76}
$$

where $d^2 \xi = d\xi_x d\xi_y$. In the small angle approximation the projected angles combine to give the relation

$$
\xi^2 = \xi_x^2 + \xi_y^2 \tag{77}
$$

The quantity

$$
\mu <\xi^2> = \int \xi^2 \mu(\xi) d^2 \xi \tag{78}
$$

is referred to as the scattering power designated T. It is conventionally defined as the rate of increase of the variance in the angular distribution with distance $z$. The quantity $<\xi^2>$ is the variance introduced in a single scatter and the total scattering co-efficient $\mu$ is equivalent to the number of scatterings per unit distance. By symmetry

$$
<\xi_x^2> = <\xi_y^2> \tag{79}
$$

so that each bracketed term on the right side of Equ.(74) is $T/4$. 
PROBLEMS FOR DISCUSSION
(RADIATIVE TRANSPORT)

1. A beam of particles moves along the x-axis in a medium with a constant stopping power of 1 keV \( \mu m^{-1} \). The incident energy spectrum, i.e. at \( x = 0 \), is given by

\[
\phi(E,0) = 5C(1.2 - E), \quad 1.0 \leq E \leq 1.2
\]

and 0 for all other energies. E is in MeV.

(a) Sketch the function \( \phi(E,x) \)
(b) Sketch the total flux density \( \phi(x) \)
Absorption is negligible.

2. A monoenergetic beam of particles moves along the x-axis in a medium with constant stopping power \( S \). The incident beam energy is \( E_0 \). The particles interact with the nuclei of the medium through an absorptive threshold reaction with interaction coefficient

\[
\mu(E) = \begin{cases} 
0 & E \leq E_t \\
= a(E - E_t) & \text{for } E > E_t 
\end{cases}
\]

Assuming \( E_0 > E_t \), find the variation of flux density with penetration, \( \phi(x) \).

3. Consider a medium in which photons only interact by absorption and elastic scattering so that \( \mu = \mu_a + \mu_s \). The differential interaction coefficient for the elastic scattering process is \( \tilde{\mu}(\Omega | \Omega') \). There is a point source at the origin, \( \tilde{Y}(\mathbf{r}, \Omega) = Y\delta(\mathbf{r})/4\pi \)

(a) Write down the Boltzmann equation for this situation.
(b) By integration over direction, find an equation relating the current density, flux density and source density.
(c) By considering a sphere of radius \( R \), find an expression for the radial component of current density, in terms of the source density and a suitable radial integral of the flux density.

4. Solve the point source transport equation for the case where

\[
\tilde{\mu}(E | E') = \mu_a \delta(E' - E) \delta(1 - \Omega' \cdot \Omega)
\]

5. Solve the problem of an infinite plane source \( \tilde{Y} \) in an infinite, homogeneous, purely absorbing medium.