SHIELDING AND DOSE CALCULATIONS

1. Point sources and infinite media

Consider the dose due to a monoenergetic photon point source imbedded in an infinite medium. The radiation field and associated dose at any point in the medium will consist of a direct contribution, \( D_d \), and a scattered contribution \( D_s \) leading to a total dose

\[
D = D_d + D_s \tag{1}
\]

Both terms in the above equation are directly dependent upon the strength of the source so that the scattered dose is implicitly dependent on the direct dose. A build-up factor is then defined through the relation

\[
D_s = (B-1)D_d \tag{2}
\]

The actual value of the build-up factor, \( B \), will depend on the amount of scattering which in turn depends upon the photon energy and the nature of the medium. In addition the build-up factor will depend upon the distance of the dose point from the source measured in mean free paths.

Taking the point source to be at the origin and to have strength \( Y(r, E) = Q\delta(r)\delta(E - E_\gamma) \), then the dose at distance \( r \) may be written

\[
D(r) = \frac{(\mu_{en}/\rho)E_\gamma Q\hat{\mu}B(E_\gamma, \hat{\mu}r)}{4\pi r^2} \exp(-\hat{\mu}r) \tag{3}
\]

where \( \mu_{en}/\rho \) is the mass energy absorption coefficient and \( \hat{\mu} \) is the linear attenuation coefficient for photons of energy \( E_\gamma \) in the medium of interest. Consider the example of a point source of 300 keV photons in water as illustrated in figure 1. Here the ordinate is the combination \( r^2D(r) \), where \( r \) is the distance from the source, scaled to unity at the origin. The abscissa is the dimensionless quantity \( \hat{\mu}r \), corresponding to the distance measured in units of the mean free path. The direct contribution is a simple exponential. Note that the effect of scattering results in an initial increase in \( r^2D(r) \) and that at large distances the contribution from scattered photons is dominant.

Figure 1. Variation of \( r^2D(r) \) with \( \hat{\mu}r \)
2. **Analytic representations**

The exact form of the build up factor is not calculable. A closed form expression would require an analytic solution to the Boltzmann equation, an impossible task with the complexity of the scattering kernel. Numerical solutions for infinitely homogeneous geometries have been attained to a good approximation. Moreover recourse has been made to Monte Carlo simulations particularly for finite geometries. These approaches however can only produce discrete values of the build-up factor in tabular form. It is convenient to approximate these tabulations by ad hoc analytical representations which can then be used in practical dose calculations and shield designs. Three common representations used are:

a) Linear form

\[ B(x) = 1 + \alpha(E, Z) x \]  

b) Berger's form

\[ B(x) = 1 + C(E, Z) x e^{D(E, Z) x} \]  

c) Taylor's form

\[ B(x) = A(E, Z) e^{-\alpha(E, Z) x} + (1 - A(E, Z)) e^{-\alpha(E, Z) x} \]  

In the above \( x \) is the dimensionless distance variable corresponding to the number of mean free paths. The linear form is not very adequate as might be expected since representation of a complex function by a single parameter is unlikely to be successful. The most popular form is that given in Equ.(6) referred to as Taylor's form. With this representation the functional relationship between dose and distance is identical to that for the direct dose. As can be seen from Equ.(3) the dose resulting from the use of this representation is equivalent to the sum of two effective direct doses. These latter result from an effective source of strength \( A(E, Z) Q \) submerged in a medium with effective attenuation coefficient \( [1 + \alpha_1(E, Z)] \mu \) and an effective source of strength \( [1 - A(E, Z)] Q \) submerged in a medium with effective attenuation coefficient \( [1 + \alpha_2(E, Z)] \mu \). Therefore if the direct dose is known in closed form for a particular geometry the effect of build-up may be incorporated using the above without recalculating the relationship as would be necessary for either of the other two representations.

In addition to the above, a general polynomial with energy and medium dependent coefficients may be used. This representation follows naturally from the moment method of numerical solution to the Boltzmann equation. Tenth order polynomials for water have been given in MIRD.

3. **General properties**

There are some simple constraints which may be imposed upon the function \( B(x) \). The first of these follows from the fact that the dose from direct radiation must become dominant at points very much less than a mean free path from the source, and is given by

\[ B(0) = 1 \]  

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and is incorporated in Equ.(4),(5) and (6). A second constraint results from energy conservation which can be written in the form

\[ \int D(r) \cdot \rho \cdot d^3r = QE \]  

where the volume element may be written

\[ d^3r = 4\pi r^2 dr \]  

From Equ.(3),(8) and (9) the constraining relation

\[ \int B(x) e^{-x} dx = \hat{\mu}/\mu_{en} \]  

is easily derived. For the linear model the above relation uniquely determines the parameter as

\[ \alpha = \frac{\mu}{\mu_{en}} - 1 \]  

4. The exponential integral functions

A set of functions which are of intrinsic mathematical interest frequently occur in dose calculations and are referred to generically as the exponential integral functions. They are defined by the equation

\[ E_n(y) = \int_{1}^{\infty} \frac{e^{-yx}}{x^n} dx \]  

The formal definition is not so useful in dose calculations. In these the integral form usually appearing satisfies

\[ \int_{a}^{\infty} \frac{e^{-yx}}{x^n} dx = \frac{1}{a^{n-1}} E_n(ya) \]  

which follows from Eqn(12) with \( x = ax' \).

The exponential integral functions satisfy the following recursion relations.

\[ E_n(y) = \int_{y}^{\infty} E_{n-1}(x) dx \]  

The above equation can also be written in the form
\[ E_n(y) = \frac{1}{n-1} \cdot e^{-y} \cdot y^{n-1} \cdot E_{n-1}(y) \]  

(15)

For \( y > 10 \) the functions may be approximated by

\[ E_n(y) \approx e^{-x} \left[ \frac{1}{x+n} + \frac{1}{(x+n)^3} \right] \]  

(16)

Tabulations of the exponential integral functions may be found in the N.B.S. handbook of mathematical functions.

5. **Dose point kernel methodology**

Dose calculations making use of point kernels rely on the linear relationship between source strength and dose. Dose point kernel methods are quite general and can be used for any type of radiation. Photon dosimetry will be emphasized in this exposition only because it is somewhat more amenable to analytical calculations. A major limitation in these calculations is the difficulty in incorporating boundary effects in general. For this reason an infinite homogeneous medium is generally assumed in what follows.

Let \( K(R) \) be the dose produced at a distance \( R \) from a point source of radiation of unit strength. This function is referred to as the dose point kernel. Consider a source which is distributed throughout a volume of space with density \( Y(r') \). At a point \( r \) the distance from the point in the source distribution is \( R = |r - r'| \) so that an element of source volume of magnitude \( d^3r' \) produces an effect the magnitude of which is \( K(R) \cdot Y(r') \cdot d^3r' \). This result rests upon the linear assumption and the fact that the distributed source may be decomposed into an infinite set of infinitesimal point sources. The dose at \( r \) may then be calculated according to the integral transform

\[ D(r) = \int K(r, r') \cdot Y(r') \cdot d^3r' \]  

(17)

where the explicit dependence of the kernel on both the source and dose positions has been indicated.

For a gamma source the dose may be calculated separately for each spectral component with effective source strength \( p(E_\gamma) \cdot Y(r) \) where \( p(E_\gamma) \) is the emission probability. It is thus sufficient to restrict consideration to monoenergetic sources. The dose point kernel then becomes

\[ K(R) = \left( \frac{\mu_{en}}{\rho} \right) E_\gamma B(\hat{\mu}R) e^{-\hat{\mu}R/4\pi R^2} \]  

(18)
In the above equation it should be realized that the dependence on energy and material of the linear attenuation and mass energy absorption coefficients as well as the build up factor have not been shown explicitly.

Specific examples of dose calculations indicate that even for very simple problems the integrals involved are not tractable. Historically the most famous example is that of the uniform line source. Such sources are used medically in the form of radium needles and the dose calculations for this case were first performed by Sievert. The geometry is indicated in Fig 2. If the source density is \( Y \) per unit length then the direct dose for a given spectral component may be written

\[
D_d = \Gamma_\delta Y \int e^{-\hat{\mu}R} \frac{dx}{R^2} \tag{19}
\]

In the above equation \( \Gamma_\delta \) is the specific dose gamma constant for the particular spectral component, given by \((\mu_e/\rho)E_\gamma/4\pi\), and \( dx \) is an element of the line source. The variables \( x \) and \( R \) may be related to the single variable \( \theta \) and the fixed perpendicular distance \( z \) by the transformations

\[
R = z \cdot \sec \theta \\
x = z \cdot \tan \theta \tag{20}
\]

Substitution leads to the resultant integral form

\[
D_d = \Gamma_\delta Y \int_0^{\theta_2} e^{-\hat{\mu}z \sec \theta} d\theta \\
= \Gamma_\delta Y \int_0^{\theta_2} [F(\theta_2, \mu z) - F(\theta_1, \mu z)] d\theta \tag{21}
\]

where the angular integration limits correspond to the ends of the line source and the lower limit is considered to be negative. The integration involved cannot be completed in closed form so that numerical evaluations must be used. These have been tabulated in the form of the Sievert integrals defined as
\[ F(y, a) = \int_{0}^{\frac{\pi}{2}} e^{-asec \theta} d\theta \] (22)

together with the convention \( F(-y, a) = -F(y, a) \).

A second simple geometry is that of a disc of uniform areal density \( Y_o \) of radius \( a \) as indicated in Fig.3. Consider a point on the axis of the disc at a distance \( z \) from the surface. If \( r \) is the usual radial coordinate in a cylindrical polar coordinate system with origin at the center of the disc, then by symmetry a suitable source element is a ring of area \( 2\pi r \) dr. The direct dose is now given by

\[ D = 2\pi \Gamma \delta Y_o \int_{0}^{a} \frac{e^{-\mu R}}{R^2} r \ dr \] (23)

We now make use of

\[ R^2 = r^2 + z^2 \] (24)

to obtain the relation

\[ RdR = r \ dr \] (25)

so that Equ(23) can be rewritten

\[ D_d = 2\pi \Gamma \delta Y_o \int_{z}^{\sqrt{z^2 + a^2}} e^{-\mu R} \frac{dR}{R} \] (26)

The limits of the integral in the above run from \( z \) to \( z sec \theta_0 = \sqrt{z^2 + a^2} \) so that

\[ Figure 3. Disc source geometry. \]
\[ D_d = 2\pi \Gamma_0 Y [E_1(\hat{\mu} z) - E_1(\hat{\mu} z \sec \theta_0)] \]  \hspace{1cm} (27) 

If Taylor's form for the build-up factor is used then the total dose may be written

\[ D = 2\pi \Gamma_0 Y [AT(\mu_1 z, \theta_0) + (1 - A)T(\mu_2 z, \theta_0)] \]  \hspace{1cm} (28) 

where

\[ T(x, \theta) = E_1(x) - E_1(x \sec \theta) \]  \hspace{1cm} (29) 

and

\[ \mu_i = (1 + \alpha_i) \hat{\mu} \]  \hspace{1cm} (30) 

6. Inhomogeneity and finite geometry

The situation often arises in which a shield consists of layers of differing materials. The question arises how best to represent build up in this situation. The shield will be considered to be prepared from N layers in which the thickness of layer j is \( t_j \) and the attenuation coefficient is \( \mu_j \) to give the dimensionless quantity \( x_j \) representing the product \( \mu_j t_j \).

In Goldstein's method the layered shield is replaced by a homogeneous shield having the same interaction probability or

\[ x = \sum_{j=1}^{N} x_j \]  \hspace{1cm} (31) 

and with an effective atomic number

\[ \bar{Z} = \frac{\sum_{j=1}^{N} x_j Z_j}{x} \]  \hspace{1cm} (32) 

corresponding to the weighted average of the individual component atomic numbers. The weighting factor is dimensionless layer thickness. The build-up factor for the entire shield is then \( B(E, \bar{Z}, x) \) and is obtained by interpolation for energy, atomic number and dimensionless distance.

Broder's method is somewhat more complicated. To see the rationale for this method consider a two-layer shield and the linear approximation. For a homogeneous shield of thickness x, decomposition into layers of thickness \( x_1 \) and \( x_2 \) is a possible if arbitrary procedure which enables one to write the build-up factor as
Equation (33) indicates that the build-up factor for the entire shield can be written as the build-up factor for the first layer to which is added a correction due to the second layer. The correction can obviously be written

\[ \alpha x_2 = B(x_1 + x_2) - B(x_1) \]  

Consider now the situation in which each layer is of a different material. Then the approach for the combined shield is to use the above with appropriate factors for each layer giving

\[ B(x_1 + x_2) = B_1(x_1) + B_2(1 + x_2) - B_2(x_1) \]  

The implication is that the added effect of the second layer is equivalent to the added effect which would have been observed in a homogeneous shield of material of type 2 when the dimensionless thickness is increased from \( x_1 \) to \( x_1 + x_2 \). In the presentation given in Equation (35) no particular form of the build-up factor is any longer assumed. This approach may be extended to the general case by the recursive relation

\[ B\left( \sum_{j=1}^{N} x_j \right) - B\left( \sum_{j=1}^{N-1} x_j \right) + B_N\left( \sum_{j=1}^{N} x_j \right) - B_N\left( \sum_{j=1}^{N-1} x_j \right) \]  

The effect of finite geometry has been investigated somewhat by Monte Carlo simulation. A correction factor corresponding to the ratio of the finite to infinite build-up factors \( B(x)/B_{\infty}(x) \) has been calculated and is given in table 1.

<table>
<thead>
<tr>
<th>Material</th>
<th>Energy (MeV)</th>
<th>Thickness (x)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Water</td>
<td>.412</td>
<td>.715</td>
</tr>
<tr>
<td></td>
<td>1.120</td>
<td>.740</td>
</tr>
<tr>
<td></td>
<td>2.100</td>
<td>.800</td>
</tr>
<tr>
<td>Iron</td>
<td>.412</td>
<td>.805</td>
</tr>
<tr>
<td></td>
<td>1.120</td>
<td>.890</td>
</tr>
<tr>
<td></td>
<td>2.100</td>
<td>.905</td>
</tr>
<tr>
<td>Lead</td>
<td>.412</td>
<td>.975</td>
</tr>
<tr>
<td></td>
<td>1.120</td>
<td>.980</td>
</tr>
<tr>
<td></td>
<td>2.100</td>
<td>.990</td>
</tr>
</tbody>
</table>

In the worst case, that of water at .412 MeV the infinite build-up factor overestimates that of a finite shield of 6 mean-free paths by a factor of about 2.5. Infinite build-up factors are normally used for finite shields, the argument being that the resultant dose estimates represent conservative design.
8. Isotope shielding factors and X-ray exposure constants

Since the attenuation and build-up factors are energy dependent the extension of point kernel methods to complex spectra, while straightforward, deserves special consideration. Consider first a radioisotope with spectrum of energy and emission probability \( \{E_j, p_j; j=1,N\} \). Then the dose produced by a point source of cumulative activity \( \tilde{A} = \lambda NT \), where \( T \) is the exposure time, through a shield of thickness \( t \) may be written

\[
D(t) = \sum_{j=1}^{N} \left[ \frac{\mu_{en}}{\rho} \right] E_j p_j B(\mu_j t) e^{-\mu_j t} \frac{\tilde{A}}{4\pi r^2} \tag{37}
\]

where \( r \) is the distance from the source. The factor in the braces, together with the \( 4\pi \) in the denominator is a gamma constant modified to take into account the effect of the shield. If this quantity is designated \( \Gamma(t) \) then a shielding factor is defined as

\[
S(t) = \frac{\Gamma(t)}{\Gamma(38)}
\]

where the denominator is the unmodified gamma constant. Graphs and tables of \( S(t) \) have been prepared for commonly used combinations such as \( ^{60}\text{Co} \) in lead.

The approach used for X-rays is slightly different. For an X-ray machine operating at current \( I \) for time \( T \) the exposure produced through a barrier of thickness \( t \) by the direct beam can be written

\[
X = K(t) \frac{IT}{R^2} \tag{39}
\]

where

\[
K(t) = \frac{1}{W} \int B(E,t)e^{-\mu(E)} \left[ \frac{\mu_{en}(E)}{\rho} \right]_{air} E \cdot n(\gamma) dE \tag{40}
\]

In the above equation \( B(E,t) \) is the build-up factor for an X-ray of energy \( E \) penetrating a material with attenuation co-efficient \( \mu(E) \) and thickness \( t \) while \( n(\gamma) \) is the number of X-rays produced per unit energy interval at \( E \) from an incident electron. Graphs of \( K(t) \) are presented for machines of various kV and for commonly used materials such as lead and concrete. In most cases the machine voltage is not constant but is rectified a.c. so that the curves are characterized by the peak anode voltage or kVp. The shield designed for the direct beam is referred to as the primary barrier.

In addition provision is made for shielding from radiation which leaks through the X-ray housing and for X-rays scattered from the patient. Because of the high degree of filtration the leakage radiation is approximated by a monoenergetic spectrum at the maximum energy. For energies which are sufficiently small compared to \( mc^2 \) Compton scattering results in a negligible energy change. For diagnostic machines the scattered spectrum is thus assumed to be identical to the primary spectrum. Thus the same \( K(t) \) curves suffice for the assessment of this component. The scattered exposure may be written
In the above expression \( a \) is a constant of proportionality, \( F \) is related to the area of the primary beam, referred to as field size, \( R_1 \) is the distance between patient and exposure point while \( R \) is the distance between the machine target and the patient. In practice \( F \) is referred to a standard of 400 cm\(^2\) and therefore becomes a dimensionless quantity. For example a value of 0.3 for \( F \) would be used for a 120 cm\(^2\) beam area. The quantity \( a \) depends upon angle and kV. For 90° it varies from 0.00035 m\(^2\) at 50 kV to 0.0019 m\(^2\) at 300 kV.

9. Internal Dosimetry

Radioisotopes may either enter the body accidentally through inhalation or ingestion, or may be administered for diagnostic procedures in nuclear medicine. While the basic principles of dosimetry remain unchanged, a specialized approach based upon reasonable assumptions and approximations has been developed. Fundamental to this approach is the assumption that the radioisotope distribution in the body may be subdivided on an organ basis, and that the distribution within each organ is uniform. The dose to any organ is the energy absorbed divided by the organ mass, and this is equivalent to the average of the dose at each point over the organ volume. The organ for which the dose rate is being assessed is called the target organ. It in general receives radiation from the radioisotope contained in each organ of the body containing the isotope including itself-this last being called the self dose. Distinct organs are generally referred to as source organs. The dose is of course the integral of the dose rate with time. The variation of the dose rate from a given source is determined by the manner in which the activity, and hence number of radioactive atoms in the source varies with time. This variation is partly determined by the radioactive decay, but since the radioisotope may be excreted, is also determined by physiological factors. This latter is generally characterized by a retention function, which describes the decrease with time for the stable isotope, as the probability of the atom remaining after time \( t \). Since the radioactive atom must both remain and survive decay, the number in the organ at any time is the initial number multiplied by the product of the retention function and the exponential decay factor, \( e^{-\lambda t} \). Often the retention function is also represented by an exponential, and characterized by a biological half-life. The time integral of the activity is called the cumulative activity and written \( \bar{A} \). For short exposure times this is the product of activity and time as assumed in the previous section.

The general situation is illustrated in Fig. 4. If the volume of the source organ is \( V_k \), then the cumulative activity density is \( \bar{A}_k/V_k \). The cumulative activity in a volume element \( d^3r_k \) is then \( \bar{A}_k d^3r_k/V_k \). If the DPK approximation is made, the element of dose produced is then the product with the dose point kernel \( K(R) \). The dose at the point indicated in the target is then

\[
X_s = K(R) \cdot \frac{aF}{R_1^2} \cdot \frac{IT}{R^2}
\]
\[ D(r_j) = \frac{\tilde{A}_k}{V_{kY_k}} \int K(R) d^3 r_k \]. Since by definition this is the ratio of the energy to the mass deposited in the volume element at \( r_j \) while the dose to the target organ, \( D_{jk} \), is the ratio of the energy to mass of the organ, or

\[ D_{jk} = \frac{1}{M_j} \int D(r_j) \rho_j d^3 r_j \] where \( \rho_j \) is the density of the target.

Combining the above with \( M_j = \rho_j V_j \) gives

\[ D_{jk} = \tilde{A}_k \int \frac{1}{V_j V_{kY_k}} \int K(R) d^3 r_k d^3 r_j \] (42)

The integral above can be written explicitly in terms of \( R \) by transforming to the variables \( r_k \) and \( R = r_k - r_j \). Using \( d^3 R = R^2 dR d^2 \Omega \)

\[ D_{jk} = \tilde{A}_k \int_{R_{\text{min}}}^{R_{\text{max}}} K(R) \left[ \frac{1}{V_j V_{kY_k}} \int R^2 d^2 \Omega d^3 r_k \right] dR = \tilde{A}_k \int_{R_{\text{min}}}^{R_{\text{max}}} K(R) p_{jk}(R) dR \] (43)

The quantity \( p_{jk}(R) \) is the point pair probability density function for \( R \). It depends on the shape and relative orientation of the two organs as well as their distance apart. The integral is designated \( S_{jk} \) and depends both upon geometry through the point pair pdf and the isotope through the dose point kernel. The dose to the target organ then may be written in the form

\[ D_j = \sum_{k=1}^{N} S_{jk} \tilde{A}_k \] (44)

An important special case in general is the term with \( j=k \), corresponding to the dose induced in an organ by the radioactivity contained therein. In this case, short-ranged charged particle radiation will contribute, unlike the case in which the source and target are separated such that \( R_{\text{min}} \) is greater than the range. In this case Eqn(43) is applied with the pdf corresponding to the manner in which the distance between pairs of points within the organ is distributed. As with all cases an exact calculation must be done numerically. However, it is usually the case that the dimensions of the organ are large with respect to the range, in which case the equilibrium dose \( \tilde{A}_j \langle E \rangle / M_j \) may be used. The average energy released in beta decay must be calculated from the beta decay spectrum. For alpha decay the spectrum is composed of discrete groups. The contribution to \( S \) is then the ratio of the average decay energy to the organ mass, which must be added to the contribution from the \( \gamma \)-radiation.

A practical complication which arises is that the description of the situation is usually incomplete in that the activity in every organ is not known. The difference between the known organ activities and the total
activity administered is the assigned arbitrarily to an effective organ, referred to as the rest of the body.

10. Neutrons

Neutrons are treated somewhat differently than photons because they produce a charged particle field with a rather complex LET distribution. For this reason estimates are usually made of the flux energy spectra and dose equivalent conversion factors used. Normally fast neutrons are produced in a reactor or accelerator in the range of several MeV. The dominant modes of interaction are elastic and inelastic scattering with the former generally being the most important. Elastic scattering from heavy nuclei has little effect on the energy. Moreover the larger the nuclear radius the greater the number of partial waves involved and the more forward directed the scattering. A forward scattering with small energy change has very little effect on the transport and so is of little use in shielding. For this reason, the use of hydrogen with its nearly isotropic scattering and light mass is commonly incorporated into neutron shields. In any event, when general materials are considered it is important to isolate those interactions which have a significant effect on the transport. Such an interaction is referred to as a removal process. Certainly all non-elastic processes such as \((n,p)\) and \((n,n'p)\) reactions are removal processes. Elastic scattering with sufficiently large angle may also contribute to removal.

Following a removal interaction the neutron is then treated as commencing the moderation process as described by a form of age-diffusion theory. A removal cross section is defined operationally by determining the attenuation in neutron flux measured in a moderating medium produced by the interposition of a thickness of the material in question. The attenuation factor \(F\) determined for thickness \(t\) is then used to determine the macroscopic removal cross section according to

\[
\Sigma_R = \ln(F)/t
\]  

(45)

The removal cross section may also be thought of as given by

\[
\Sigma_R = \Sigma_{ne} + f\Sigma_s
\]  

(46)

where \(\Sigma_{ne}\) is the non-elastic cross section, \(\Sigma_s\) is the elastic scattering cross section and \(f\) is the fraction of scatterings with sufficiently large angle. Admittedly this latter is rather vague but can be chosen for consistency with the value in Eqn.(45). It is found in practice generally that the removal cross section is approximately given by

\[
\Sigma_R \approx \frac{2}{3}\Sigma_t
\]  

(47)

where \(\Sigma_t\) is the total cross section. For water the removal cross section is taken to be the total cross section.

Various neutron point kernels have been proposed for calculating the unremoved flux density from a distributed source. These are based on exponential forms. For example if a source is given by a spectral distribution \(p(E)\) the point kernel is of the form.
\[ K(R) = \frac{1}{4 \pi R^2} \int p(E) \exp(-\Sigma_r(E)R) dE \]  \hspace{1cm} (48)

More convenient is the use of forms such as

\[ K(R) = \frac{1}{4 \pi R^2} \exp(-\Sigma_r R) \]  \hspace{1cm} (49)

in which a single average removal cross section is used. For water a dose point kernel of the more general form

\[ K(R) = \frac{1}{4 \pi R^2} \sum_{i=1}^{n} A_i e^{-\alpha_i R} \]  \hspace{1cm} (50)

is used. When a mixed shield consisting of water and a material with removal cross section \( \Sigma_r \) is used the co-efficient of \( R \) in the exponent is replaced by \( g_w \alpha_i + (1-g_w) \Sigma_r \) where \( g_w \) is the fraction by weight of the water.

If the unremoved fluence rate calculated using the appropriate point kernel is \( \varphi_u(r) \) then the rate of removal interactions per unit time and volume is \( \Sigma_r \varphi_u(r) \). This rate acts as a source for the age-diffusion equations. Typical parameter values for such calculations are given in table 2.

<table>
<thead>
<tr>
<th>Material</th>
<th>( \Sigma_r )(cm(^{-1}))</th>
<th>( \tau )(cm(^2))</th>
<th>( L_d )(cm)</th>
</tr>
</thead>
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<tr>
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<td>2.76</td>
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<tr>
<td>Fe</td>
<td>.166</td>
<td>200</td>
<td>1.33</td>
</tr>
<tr>
<td>Pb</td>
<td>.118</td>
<td>5000</td>
<td>13.60</td>
</tr>
</tbody>
</table>

Numerical calculation of the age diffusion process may be carried out by quantizing the energy variable into discrete energy bins or groups. This process, known as multi-group diffusion, leads to a series of coupled equations in the following manner.

The general diffusion equation with slowing down in a source-free region may be written

\[ D \nabla^2 \varphi - \sum_{a} \varphi = -\frac{\partial (S \varphi)}{\partial E} \]  \hspace{1cm} (51)

Energy space is now quantized into finite groups of width \( \Delta E \) so that the energy of each group may be written

\[ E_i = E_0 - i \Delta E \]  \hspace{1cm} (52)
Letting $\phi(E_i) = \phi_i$ be the fluence rate of group $i$ the slowing down term can be approximated by

$$\frac{\partial \phi(E)}{\partial E} = (\Delta E)^{-1}[\phi(E+\Delta E)S(E+\Delta E)-\phi(E)S(E)]$$

$$= (\Delta E)^{-1}[\phi_{i-1}S_{i-1}-\phi_S]$$

$$= D_{i-1}\phi_{i-1}/\tau_{i-1} - D_i\phi_i/\tau_i$$

(53)

In obtaining this result the finite approximation to the age required to slow between successive groups, $\tau = D\Delta E/S$, has been used. The equation for a given group thus becomes

$$D_i\nabla^2\phi_i - (\sum a_i + D_i/\tau_i)\phi_i = -D_{i-1}\phi_{i-1}/\tau_{i-1}$$

(54)

These equations hold for $i>1$. In the equation for the first group the right hand side is replaced by $-\sum a_i\phi_u$, the removal interaction rate.

11. **Accelerator shielding**

The advent of very high-energy machines has lead to unique shielding problems not widely encountered. Since the energies reached are approaching the cosmic-ray energy range, cascade phenomena occur. A very high-energy proton will produce one or more high-energy secondary nucleons in interacting with a target nucleus. These secondaries in turn may breed an even larger number of tertiary nucleons and so on. The initial energy concentrated on the single incident proton is distributed between the generated particles and target excitation. The multiplication process ceases for those generations with average energy below the threshold for multiple nucleon production.

Such a process is referred to as a hadronic cascade. Similarly high energy electrons produce high energy photons by radiative stopping which in turn produce leptons by pair production resulting in an electromagnetic cascade. It is interesting to note that these two types of cascade are in fact coupled to some extent. Thus photoneutron reactions may produce a hadronic cascade initiated during an electromagnetic cascade. Similarly high-energy photons generated in a hadronic cascade can initiate an electromagnetic cascade. Discussion here will be restricted to a highly oversimplified analysis of the hadronic cascade.

The multiplicative hadronic interaction is represented by a removal cross section $\Sigma_a$ and a multiplication $m$. The straight-ahead approximation is used. It has been found empirically that the removal cross section is essentially constant for $E > 150$ MeV. Below this region multiplication ceases but the removal cross section rises dramatically. Thus the mean free path of nucleons with energies less than 150 MeV is much shorter than for the higher energy nucleons. The nucleons are thus divided into two groups by this threshold referred to as the propagating group ($E>150$ MeV) and the non-propagators. The latter are then taken to have the same spatial dependance as the propagators.

The equation for successive propagator generations may be written

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\[
\frac{d\phi_{i+1}}{dx} + \sum_R \phi_{i+1} = m_i \sum_R \phi_i
\]  \hspace{1cm} (55)

The equation for the first generation is
\[
\frac{d\phi_1}{dx} + \sum_R \phi_1 = m_1 \sum_R \phi_u(0) e^{-\sum_R x}
\]  \hspace{1cm} (56)

where \(\phi_u(0)\) is the incident primary (unremoved) fluence rate. The solution to the above is
\[
\phi_1(x) = m_1 \sum_R \phi_u(0) xe^{-\sum_R x}
\]  \hspace{1cm} (57)

The solution for generation \(n\) is obtained by successive solution of Equ.(55) beginning with \(i=1\) and Equ(57). The result is
\[
\phi_n(x) = \frac{1}{n!} \prod_{i=1}^{n} m_i \phi_u(0)(\sum_R x)^n e^{-\sum_R x}
\]  \hspace{1cm} (58)

The total fluence rate in the cascade is then
\[
\phi(x) = \sum_{n=0}^{n_{\max}} \phi_n(x) = B(x) \phi_u(0) e^{-\sum_R x}
\]  \hspace{1cm} (59)

where \(B(x)\) may be considered as a flux density build-up factor.

An important property of functions of the form \(x^n e^{-\sum_R x}\) is that for large distances they approach the exponential. This can be seen by considering the behaviour of the logarithmic derivative
\[
\frac{d}{dx} \ln(x^n e^{-\sum_R x}) = \frac{n}{x} \sum_R
\]  \hspace{1cm} (60)

The first term in the above decreases with distance so that deep in the shield the build-up factor becomes independent of \(x\).

These considerations were used by Moyer to develop the following model for the dose equivalent produced by a high energy beam. If the beam current is \(I\) then the production rate of primaries with energy \(E\) moving in direction \(\Omega\) due to interactions with the target may be written
\[
n(E,\Omega) = (I/e) n_{\gamma} \frac{d^2\sigma}{dEd\Omega}
\]  \hspace{1cm} (61)
As these primaries move through the shield they initiate a cascade. The propagators of the cascade are accompanied by short range non-propagating secondaries. Then the dose equivalent rate at a distance \( r \) from the target takes the form

\[
\dot{H} = F \cdot \int_{E_{m}}^{E_{m}} B(E) n(E, \Omega) dE \cdot \frac{\exp(-\sum d)}{r^2}
\]  \( (62) \)

Here \( F \) is the dose equivalent conversion factor, \( d \) is the path length through the shield and \( E_{m} \) is the maximum energy. It should be emphasized that the LET structure of the cascade is of course complicated so that estimation of the appropriate quality factor which determines \( F \) is extremely difficult. The fluence build up factor \( B(E) \) has been taken in the deep shield approximation and its dependence on the primary energy is indicated explicitly.

The integral in the above equation is empirically found to vary exponentially with the complementary angle to the angle between the beam direction and the direction of propagation and is represented by \( g(E_{m}, \theta) \) so that

\[
g(E_{m}, \theta) = m(E_{m}) e^{-\beta(E_{m}) \theta}
\]  \( (63) \)

The geometry is indicated in Fig.5. The average multiplicity in Equ. (53) is absorbed into a new constant with \( F \) to give the three-parameter representation

\[
\dot{H} = \dot{H}_{0}(E_{m}) e^{-\beta(E_{m}) \theta} \cdot \frac{\exp(-\sum d_{0} \sec \theta)}{r^2}
\]  \( (64) \)

The three parameters are determined empirically for existing machines. More sophisticated calculations such as Monte Carlo simulations may be presented in terms of these three parameters for comparison. Analysis of the trends with energy are then used to extrapolate shield design for planned higher energy machines.

**Figure 5.** Moyer model geometry
1. Calculate the direct dose from a uniform disc of radioactivity at a distance $z$ on the axis through a shield of thickness $t$. The shield attenuation coefficient is $\mu$, the activity per unit area is $s$ and the dose specific gamma constant is $\Gamma$.

2. A layered shield of water and iron consists of 3 mfp of water followed by 2 mfp of iron. The build up factor at the energy of interest is 4.5 at 3 mfp in water. It increases from 2.2 at 3 mfp to 4.3 at 5 mfp in iron. Calculate the build-up factor for the shield.

3. Find the solution to the multi group diffusion equation in one dimension in a source free region as would occur for a delta function source.